ELASTIC PROPERTIES OF THE EMPTY SKELETON IN A GRANULAR COLLECTOR

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Up to now, problems on elastic-wave propagation in granular skeletons have not been solved in rigorous formulation. This is due to difficulties in incorporating the boundary conditions at the entire complicated pore surfaces. Many researchers such as MacKenzie and Toksoz have sought to avoid this difficulty (in the case of isolated pores) by solving the exact problem for an isolated inclusion and assuming that the pores do not influence one another [1]. This formulation gives correct results only for very low porosity. In the case of arbitrary (not small) porosity, the solutions produce a fall in the Lamé coefficients λ and μ, but because of the fall in the rock density the fall in the velocities is very slight, and in particular the push-wave speeds in the empty skeleton are higher than those in a water-saturated one for a spherical shape for the pores, which conflicts with experiment [2]. Another group of researchers has been concerned mainly with composites, and they have developed approximate methods for constructing the so-called Hashin—Shtrikman fork [3, 4], where instead of the exact solution one defines acceptable boundaries to the variations in λ and μ. If the fork is sufficiently narrow, the values of λ and μ for a composite are virtually reliable. The width of the fork increases with the difference in elastic properties between the skeleton and the fluid, and estimates of the Hashin—Shtrikman type for real collectors are not acceptable on account of the large changes in λ and μ (if the pores are filled with gas) or in μ (if the pores are filled with liquid). Attempts have been made [5, 6] at rigorous calculation of the elastic properties for empty skeletons with periodic structures. Such calculations can be performed because the boundary conditions in that case are posed at the boundary of a single unique period, and then these conditions are periodically repeated in the microstructure. The problem then amounts to determining the average stress and strain tensors for some period in the structure. An algorithm has been given for obtaining the exact solution [6], and theory and experiment are found to be in very good agreement for two-dimensional media.

However, a periodic structure is a particular case of a microstructure. In fact, there is periodic repetition only for certain integral characteristics of the microstructure, but not for the individual properties. At the same time, it is virtually impossible to indicate a microstructure of sufficiently small period if the structure is composed of particles differing in size, as occurs in terrigenous collectors. Finally, all periodic structures form anisotropic bodies. One can obtain an isotropic and statistically homogeneous body only on nonperiodic structures with a fairly random organization. Therefore, one needs a new approach to determining the mean values of λ and μ for microscopically inhomogeneous and statistically isotropic media.

The following arguments can be advanced here. One can consider the rigorous solution of the equilibrium equations for an individual grain having a certain number of areas of contact with its neighbors. This problem can be solved if the forces at the contacts are given. However, one can assume that these forces are undetermined on the basis that the center of gravity of the grain is deformed as in a planar longitudinal wave, i.e., $e_{xx} = e_{yy} = 0$, $e_{zz} \neq 0$, $\sigma_z = 1$ at the grain center. If one considers the stresses on the grain as balanced (which is reasonable because the grain radius is much less than the wavelength) and the deformation energy is assumed minimal, then the problem becomes completely defined and the forces can be calculated as functions of the grain material and structure of the pore space under these conditions. From these forces one can determine the average strain tensor at the center of the grain and the average stress tensor. The relation between these gives the mean values of the Lamé parameters λ and μ for any grain and therefore for the entire microstructure.

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We consider the situation where the wavelength is much greater than the particle radius r_0, i.e., $l \gg r_0$; in that case, the inertial forces are very small by comparison with those produced by the internal stresses.

The expressions are as follows. The bulk forces set up by the stresses, i.e., div σ_{ik} or order σ_{ik}/r_0, vary substantially over distances of the order of the grain size, since div $\sigma_{ik} \sim \rho V_p^2 \theta / r_0$, where θ is the bulk strain. At the same time, the inertial forces are $\rho \ddot{u}_i \sim \rho u_i / T^2$, and therefore we have the condition $\rho V_p^2 \theta / r_0 \gg \rho u / T^2$, where T is the wave period. As $TV_p \sim l$ (wavelength) and $u/l \sim \theta$, we get $\theta / r_0 \gg \theta / l$, and this inequality is obeyed by virtue of the condition $l \gg r_0$, and this implies the inequality $\partial \sigma_{ik} / \partial x_k \gg \rho \ddot{u}_i$. The smallness of the inertial forces by comparison with the forces set up by the internal stresses enables one to integrate the equation of equilibrium instead of the equations of motion, i.e., to determine the stress and strain fields we neglect the wave processes within a grain.

We therefore have to integrate the equation of equilibrium $\partial \sigma_{ik} / \partial x_k = 0$ in the internal region of the grain V_0 if forces fi are given at the area of contact and there are no loads over the rest of the grain surface. We construct the solution with as yet undetermined forces f_i but with a given system of areas of contact by boundary integral equation methods, with the displacement vector $u_k(x)$ represented in the form $u_k(x) = \int \varphi_i(y) \Gamma_{ik}(x, y) ds_y$, while $\varphi_k(x)$ satisfies a system of integral equations [7]:

$$\varphi_k(x) + 2 \int \Gamma_{ik}^{(1)}(x, y) \varphi_i(y) \, ds_y = f_k(x). \tag{1}$$

System (1) is singular because of the unintegrable singularity in the tensor $\Gamma_{ik}^{(1)}$ when points x and y coincide. We use a standard technique [7] to eliminate this singularity, which is based on using instead of the operator $\Gamma_{ik}^{(1)}$ the conjugate operator $\Gamma_{ik}^{(1,2)}(x, y)$:

$$\varphi_k(x) + \int_s \left[\Gamma_{ik}^{(1)}(x, y) \varphi_i(y) + \Gamma_{ik}^{(1,2)}(x, y) \varphi_i(x) \right] ds_y = \frac{f_k(x)}{2}. \tag{2}$$

System (2) does not have singularities for $x \to y$ and can be solved by successive approximation. In that method, we put $(1/2) f_k(x) = a_{km} \Omega(x_m)$, where a_{km} are constant coefficients and $\Omega(x)$ is the characteristic function of the area ds within which lies point x, i.e., $\Omega(x) = 0$ if $x \notin ds$ and $\Omega(x) = 1$ if $x \in ds$. Then the field $\varphi_k(x)$ is represented by the sum

$$\varphi_k(x) = \varphi_k^{(0)}(x) + \varphi_k^{(1)}(x) + \ldots + \varphi_k^{(n)}(x) + \ldots, \quad \varphi_k^{(0)} = \frac{f_k}{2},$$

and recurrence relations exist between the different functions:

$$-\varphi_k^{(n)}(x) = \int_s \left[\Gamma_{ik}^{(1)}(x, y) \varphi_i^{(n-1)}(y) + \Gamma_{ik}^{(1,2)}(x, y) \varphi_i^{(n-1)}(x) \right] ds_y. \tag{3}$$

Then there is a linear relationship between the displacements at a certain point x_m and the forces a_{in} at all the other points (including the point x_m). This relationship is expressed via 3n undefined constants and must be supplemented with certain conditions. To eliminate the indeterminacy in relation to the given forces a_{km}, we proceed as follows. Clearly, the actual deformation process is such that its potential energy is minimal. In fact, this assertion must apply to a volume containing a set of particles. In principle, one could apply the same scheme of arguments and calculations for several grains and minimize their potential energy. For a large number of grains, this technique might lead to solution of the equilibrium problem for a granular body. The present study does not deal with this large problem. However, it is clearly reasonable to begin such a study with a volume dV containing only one grain typical in a certain sense. The potential energy is the product of force and displacement, which is related to the force by (2), so $E = \int_s P_i u_i ds$. The displacements are linearly related to the forces, so the deformation energy is a quadratic function of the coefficients a_{km}:

$$E = u_{hm} a_{hm} = a_{ij} a_{hm} Q_{ih}^{j}(x_m, x_j), \tag{4}$$

where

$$Q_{ik}^{j}(x_m, x_j) = \Pi_{ik}^{j}(x_m, x_j) + G_{il}(x_j) \Gamma_{hl}(x_m, x_j) \Delta s.$$

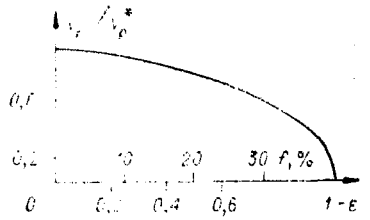

Fig. 1

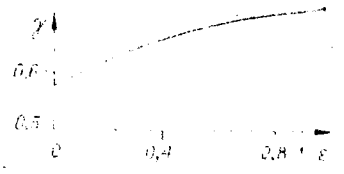

Fig. 2

In (4) we have omitted the sign of summation over the subscripts i, j, k, and m. The functions H^j_{ik}, $G_{i\zeta}$ are related to realization of the successive approximations in (3), while $\Gamma_{k\zeta}(x_m, x_j)$ is the Green's tensor for the elastic-equilibrium equation. If we sum with respect to subscript m, which runs through all points on the areas of contact, then

$$E = a_{ij}a_{k1}H^{j_1}_{ik}(x_1, x_j) + a_{i1}a_{k1}P_{ik}(x_1) + a_{ij}a_{k2}H^{j_2}_{ik}(x_2, x_j) + a_{i2}a_{k2}P_{ik}(x_2) + \ldots + a_{ij}a_{kn}H^{j_n}_{ik}(x_n, x_j) + a_{in}a_{kn}P_{ik}(x_n). \tag{5}$$

In (5), the symbol $P_{ik}(x_s)$ denotes $G_{i\zeta}(x_s)\Gamma_{\zeta k}(x_m, x_s)\Delta s = G_{i\zeta}(x_s)M_{\zeta k}(x_s)$, with $M_{\zeta k}(x_s)$ determined from the formula $M_{\zeta k}(x_s) = \lim_{m \to s} \Gamma_{\zeta k}(x_m, x_s)$; as regards the conditions under which (5) is minimized, these include the equilibrium conditions for the grain as a whole (not only for small volumes in the inner part). The equilibrium conditions for the grain as a whole require the vector for the surfaces to be zero and the same for the antisymmetric tensor for the principal moment of the grain:

$$\sum_{j=1}^{n} a_{kj} = 0, \quad k = 1, 2, 3; \tag{6}$$

$$\sum_{j=1}^{n} (y_j a_{j3} - z_j a_{j2}) = 0, \quad \sum_{j=1}^{n} (z_j a_{j1} - x_j a_{j3}) = 0, \quad \sum_{j=1}^{n} (x_j a_{j2} - y_j a_{j1}) = 0. \tag{7}$$

In (7), x, y, and z are the Cartesian coordinates of the contact elements. The three conditions (6) reduce the force vector to zero, while the conditions of (7) reduce the rotation moment of the grain to zero. We further assume that there are no field fluctuations at the center of the grain, i.e., the center deforms as in a planar longitudinal wave, so at the center

$$e_{11} = e_{22} = e_{12} = e_{13} = e_{23} = 0, \quad e_{33} = e, \tag{8}$$

and the deformation e differs from zero, for example e =1. Then to the six conditions of (6) and (7) we add six conditions imposed on the strains. These conditions also consist of six linear equations for the forces, as is clear from (1). The twelve conditions (6)-(8) go with the requirement of minimum strain energy to provide sufficient equations for defining the unknown forces on the areas of contact when a longitudinal wave on average planar is incident. Lagrange's method amounts to solving a system of equations of dimensions 3n +p consisting of expressions of the type

$$\frac{\partial E}{\partial a_s} + \lambda_1 \frac{\partial \varphi_1}{\partial a_s} + \lambda_2 \frac{\partial \varphi_2}{\partial a_s} + \ldots + \lambda_p \frac{\partial \varphi_p}{\partial a_s} = 0. \tag{9}$$

To equations (9) one adds p equations (in our case 12 of them) as in (7)-(9).

As E is a quadratic function of the variables a_s, and the coupling equations are linear, the problem as a whole amounts to solving a linear system of dimensions 3n +p. The expression for $\partial E/\partial a_{sk}$ takes the form

$$\frac{\partial E}{\partial a_{ks}} = \sum_{i=1}^{3} a_{is}(1 + \delta_{ik})P_{ik}(x_s) + \sum_{i,j,s} a_{ij}H^j_{ik}(x_s, x_j). \tag{10}$$

In (10), ks =1, 2,..., 3n; i, k =1, 2, 3.

As we know the forces P_i, one can determine the average stress tensor at the grain center from the following formula [8]:

$$\bar{\sigma}_{ik} = \frac{1}{2\Gamma}\int_s (P_i x_k + P_k x_i)\, ds, \tag{11}$$

where x_i, x_k are the Cartesian coordinates of the forces and V is the grain volume. The relations between these strains and the average stresses are given by the average values of the Lamé coefficients λ and μ for the entire structure. The integration in (11) is extended only to the contact region $S\varepsilon$, where ε is the fraction of the contact area in relation to the entire grain surface, so it is clear that the average stress tensor will be dependent on ε. As ε is expressed via $\eta = \sigma_0 r_0/3$ ($\sigma_0 r_0$ is the product of the specific surface and the mean grain size) in the form $1 - \varepsilon = \eta/(1 - f)$, where f is the porosity, the mean values of λ and μ for the structure will be determined by at least two geometrical parameters: the porosity f and the product of the specific surface by the mean grain size η. It can be shown that $0 \leqslant \eta \leqslant 1 - f$. Calculations have been performed for a grain with eight identical arbitrarily disposed contact areas for various values of the porosity and specific surface of the pores, which has shown that the wave speed in fact is dependent at least on two geometrical parameters. The contact areas are split up into elementary ones. The forces are taken as constant within an element. In practical calculations, one uses a grain with a spherical shape for the free part of the surface and with circular contact areas (the boundary conditions are satisfied at 720 points on the grain surface). In experiments with artificial models for granular media, it is possible to make specimens identical in porosity but with different specific surfaces. For large values of η (with a fixed porosity), the push-wave speeds fall, while the ratio $\gamma = V_S/V_p$ increases. Figures 1 and 2 show these calculations on the velocities V_p and the ratio $\gamma = V_S/V_p$ for artificial granular media with $1 - \varepsilon$ as the variable, i.e., as functions of f and η. The agreement with experiment is good. For large values of η, Poisson's ratio tends to zero and may take negative values. The density of the elastic energy at the grain center is due only to the bulk strain e_{33}, so this corresponds to the large-scale field. At other points in the grain clearly there are tangential stresses and strains. Here the fluctuation field will be represented as an integral over the grain volume:

$$\tilde{E} = \int_V \sigma_{ij} e_{ij} dV,$$

where i and j take all values apart from the value $\sigma_{33}e_{33}$, which defines the energy of the average large-scale field, i.e., $\bar{E} = \int_V \sigma_{33} e_{33} dV$. The solution enables one to calculate the important scattering parameter $\alpha = \tilde{E}/\bar{E}$ for an individual grain. We have thus constructed a numerical procedure for determining the velocities and scattering parameters of push and "shake" waves in granular media in relation to the grain material and the structure of the pore space.

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